On the Use of the Integral Method for Flow of Power-Law Fluids

MOHAMED EL DEFRAWI and BRUCE A. FINLAYSON

Department of Chemical Engineering
University of Washington, Seattle, Washington 98195

The importance of non-Newtonian fluids has motivated the investigation of flow problems involving such fluids. One common method of analysis—the von-Kármán-Pohlhausen integral method—is sometimes applied to predict the velocity profile, drag or heat transfer from flat plates, spheres, etc. (See (*1, 3, 5*) for applications of the integral method to power-law fluids and (*6*) for the applicability of the power-law model.) When this method is applied to boundary layer flow, for example, the differential equation is integrated over the boundary layer and this integral equation is satisfied by the approximate solution. To apply the method it is necessary to assume a shape for the velocity profile and the integral momentum balance gives the thickness of the boundary layer. It is known for flow of a power-law fluid past a flat plate that this procedure gives a poor prediction of the velocity for small n (~ 0.1), although the wall shear stress is predicted within 11% for $n = 0.1(1)$. This is unfortunate, because the integral method is easy to apply and better results are desirable. We show below the reason for the poor prediction of velocity: a velocity profile which is appropriate for a Newtonian fluid is not necessarily appropriate for a non-Newtonian fluid. Furthermore, we provide a method for testing different velocity profiles to choose the best one without knowledge of the exact solution. We concentrate on the flow of a power-law fluid past a flat plate since a large body of knowledge exists about that problem.

THEORY

The momentum and continuity equations are Equations (1) and (2) of (*1*). The velocity can be represented through a similarity transformation

$$u(x, y) = U_\infty \phi(\eta) \tag{1}$$

$$\eta = y/\delta(x) \tag{2}$$

The exact solution is valid for η from 0 (the wall) to ∞ (far into the main stream). It is the usual custom, however, to define a boundary layer thickness as that distance (in y) necessary for the velocity to reach 98% or 99% of its mainstream value. In the integral method we take η from 0 to 1, and then $\delta(x)$ represents the boundary layer thickness. The boundary conditions on ϕ are then

$$\phi(0) = 0, \quad \phi(1) = 1, \quad \phi'(1) = 0 \tag{3}$$

where the last condition is required if the shear rate is continuous at the edge of the boundary layer. To apply the integral method we must choose a function $\phi(\eta)$ satisfying Equation (3) (as well as possibly other conditions), substitute the velocity Equation (1) into the momentum equation, integrate from $y = 0$ to $y = \delta$ and solve for $\delta(x)$. This leads to

$$\delta(x) = (U_\infty^{n-2}\, m\, \alpha\, x/\rho)^{1/(n+1)} \tag{4}$$

where α is a separation constant determined from the integrated momentum equation

$$\int_0^1 R(\eta)\, d\eta = 0 \tag{5}$$

where the residual $R(\eta)$ is defined as

$$R(\eta) = \alpha\phi' \int_0^\eta \phi(\eta)\, d\eta + n(n+1)\,(\phi')^{n-1}\,\phi'' \tag{6}$$

To begin the analysis we must choose the function $\phi(\eta)$. As a guide to our choice we consider a related problem: flow between two flat plates. The exact solution is known (*6*)

$$u(z) = u(0)\,[1 - z^{(n+1)/n}] \tag{7}$$

where $z = 0$ is the center of the tube and $z = \pm 1$ are the walls. For n small this profile is very flat in the center of the duct and falls sharply to zero near the wall. Consider next the entry-length problem, with a uniform flow outside the parallel plate duct. As the fluid moves into the duct, a boundary layer begins to form at each wall, and this layer grows until the boundary layers from each side join in the center (see Figure 1). The velocity profile is then given by Equation (7). This description is only approximate, since the concept of a boundary layer is ambiguous. The description applies to the representation assumed when the integral method is employed to solve for the flow in the entry-length region. When solving for the flow in one boundary layer the existence of the boundary layer on the other wall is immaterial until the boundary layers join at the center. Furthermore, the velocity profile is similar so that the shape of Equation (7) applies at any length down the duct with the coordinate changed to go from the wall to the edge of the boundary layer. It is also clear that until the boundary layers join, the entry-length problem is similar to the problem of flow past a flat plate. Thus a velocity profile with the shape of Equation (7) is the proper one to use in the boundary layer analysis. We emphasize again that these ideas pertain only to the integral method applied to the problem, not to the exact solution, because the exact solution has no real boundary layer thickness.

Thus we choose the following velocity profile

$$\phi(\eta) = 1 - (1 - \eta)^{(n+1)/n} \tag{8}$$

By contrast, Acrivos et al. (*1*) used

$$\phi(\eta) = \frac{3}{2}\eta - \frac{1}{2}\eta^3 \tag{9}$$

Correspondence concerning this communication should be addressed to B. A. Finlayson.

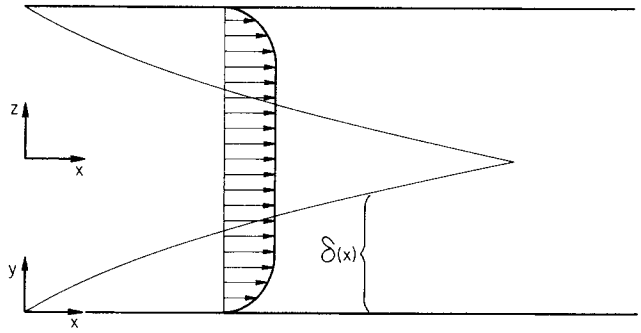

Fig. 1. Flow in the entrance region of a flat duct.

which is the standard profile for Newtonian fluids. In comparison, for small n, Equation (8) is a very flat profile, being constant near $\eta \sim 1$ and approaching zero sharply at $\eta = 0$, whereas Equation (9) is more nearly parabolic in nature. For reasons given above, particularly Equation (7), we expect Equation (8) to be the better representation.

In addition to the physical reasons for preferring Equation (8) we wish to give a mathematical reason, too. One disadvantage of approximate solutions is that the error is hard to assess. If one has a computer available, successive approximations can be calculated until the solution converges. However, then one advantage of the approximate method—its simplicity—is lost. We would like to assess the accuracy without calculating successive approximations. To do this we can use the concept of the residual.

Once a $\phi(\eta)$ is chosen and α is found from Equation (5), we can regard the residual $R(\eta)$ in Equation (6) as a known function of η. If we have found the exact solution then $R(\eta) = 0$ for all η. For an approximate solution, however, the $R(\eta)$ is not zero everywhere, and for a first approximation with the integral method $R(\eta) = 0$ usually at only one value of η. The places where $R(\eta) \neq 0$ indicates something about the error in the approximate solution, since the residual is zero everywhere for the exact solution. From experience we know that successive approximations can make the residual smaller and smaller. Thus we are led to the possibility of using the residual as an indicator of the accuracy of the approximate solution. We offer the criterion: the solutions with the smallest residuals are best.

This criterion can be made more precise for many problems. Ferguson (see (4) and forthcoming publications) has shown for problems in chemical reaction engineering that the mean square error in the solution is bounded by the mean square residual. As the residual is decreased in successive approximations, the error bound decreases as well. Here such theorems have not been proved. We accept the principle, however, as a conjecture awaiting proof. With such a criterion we can then test the trial functions, Equations (8) and (9), to see which gives the smallest residual.

The mean square residual we define as

$$|| R ||_1 = \left\{ \alpha^{-(2n+1)/(n+1)} \int_0^1 R^2(\eta) d\eta \right\}^{1/2} \qquad (10)$$

and we also define another norm

$$|| R ||_2 = \alpha^{-n/(n+1)} \int_0^1 |R(\eta)| \, d\eta \qquad (11)$$

Both norms are derived by considering the original momentum balance, integrating from $y = 0$ to $y = \infty$, with the differential equation zero for $y \geq \delta(x)$. The incorporation of α is necessary since different solutions give different α, hence different η. We are comparing the solutions on the y-axis rather than an η-axis to assure their equivalence.

One additional feature is important. In finding the exact solution after applying the similarity transformation, we obtain the differential equation $R(\eta) = 0$ with R as given in Equation (6). If we regard this as the differential equation to be solved, we can multiply by $(\phi')^{1-n}$ to remove the singularity at large η. Then we wish to solve

$$(\phi')^{1-n} R(\eta) = \alpha(\phi')^{2-n} \int_0^\eta \phi d\eta + n(n+1)\phi'' = 0 \qquad (12)$$

The integral method can be applied to this equation too, choosing α from

$$\int_0^1 (\phi')^{1-n} R(\eta) d\eta = 0 \qquad (13)$$

If this is done the results do not correspond to an integral momentum balance, due to the weighting factor $(\phi')^{1-n}$. We give solutions derived in this way, but always use the error criteria Equation (10) or (11) regardless of whether α is determined by Equation (5) or (13).

RESULTS AND DISCUSSION

We have four different approximate solutions to choose from since we have two choices of $\phi(\eta)$, Equations (8) and (9), and two integral equations, Equations (5) and (13). The choice of Equations (5) and (9) corresponds to the approach usually taken for a Newtonian fluid and the one presented for a power-law fluid by Acrivos et al. (1). Values of α are presented in Table 1. To decide which

Table 1. Properties of Approximate Solution

Velocity profile: $\phi = \frac{3}{2}\eta - \frac{1}{2}\eta^3$ $\quad \phi = 1 - (1-\eta)^{(n+1)/n}$

Differential equation	(5)	(13)	(5)	(13)
n	Values of separation constant α			
	Usual analysis			Best result*
0.1	8.24	1.35	35.1	15.5
0.2	9.36	2.93	26.0	17.8
0.5	13.2	8.87	24.3	23.1
1.0	21.6	21.6	30.0	30.0
n	Values of norm $\|\| R \|\|_1$			
	Usual analysis			Best result*
0.1	∞	∞	0.29	0.27
0.2	∞	∞	0.26	0.24
0.5	0.45	0.56	0.21	0.20
1.0	0.16	0.16	0.18	0.18
n	Values of norm $\|\| R \|\|_2$			
	Usual analysis			Best result*
0.1	1.5	1.1	1.2	0.87
0.2	1.2	0.97	0.89	0.73
0.5	0.67	0.59	0.53	0.51
1.0	0.24	0.24	0.38	0.38

* The term best result applies for small n, not $n = 1$.

approximate solution is best, without reference to the exact solution, we refer to values of $||R||_1$ in Table 1. There we see that for small n when using the new profile Equation (8) the residual is smaller than when using the usual profile and this difference is dramatic for $n < 0.5$. For $n = 1$, corresponding to a Newtonian fluid, the usual profile has a slightly smaller residual. Since some of the values of $||R||_1$ are unbounded it is interesting to make the same comparison for $||R||_2$ (see Table 1). There the same conclusion is reached, although the differences are less dramatic. For small n the new profile has the smallest residual, and the best results are achieved with the new Equation (13) rather than the usual Equation (5).

Turn now to Table 2. The $C(n)$ is proportional to the wall shear stress (*1*).

$$C(n) = \alpha^{-n/(n+1)} [\phi'(0)]^n \qquad (14)$$

The error criterion suggests that the best results for small n are obtained when the new profile [Equation (8)] is used in the new equation [Equation (13)]. For $n = 1$, on the other hand, both equations are equivalent and the best profile is the usual one Equation (9). The approximations to $C(n)$ confirm these results. For small n the new profile gives the best results when using the same equation, whereas the new equation gives best results when using the same profile. For $n = 1$ the usual profile gives better results. The same conclusions are evident in Figure 2, which shows the velocity profile for the various solutions when $n = 0.1$.

Acrivos et al. (*1*) discuss heat transfer to power-law fluids in boundary layer flow past flat plates. They show that for fluids with small n when x is small the momentum boundary layer thickness is less than the thermal boundary layer thickness. Under those conditions the approximation $u \sim U_\infty$ is valid, and the heat transfer results are independent of the velocity profile. For larger x, however, the reverse situation holds and the velocity can be approximated by

$$\frac{u}{U_\infty} = \left(\frac{U_\infty^{2-n}\rho}{m}\right)^{1/(n+1)} [C(n)]^{\frac{1}{n}} x^{-1/(n+1)} y \qquad (15)$$

The heat transfer then depends on the slope parameter $[C(n)]^{1/n}$. The usual profile [Equations (5) and (9)] has an error in this slope of from 19% to 70% as n ranges from 0.5 to 0.1. The new profile [Equations (13) and (8)] reduces this error to 8% to 25% for $0.1 \leq n \leq 0.5$. Thus the new profile is a better choice for heat transfer studies.

TABLE 2. VALUES OF WALL SHEAR STRESS PARAMETER $C(n)$

Velocity profile	$\phi = \frac{3}{2}\eta - \frac{1}{2}\eta^3$		$\phi = 1 - (1-\eta)^{(n+1)/n}$		
Differential equation	(5)	(13)	(5)	(13)	
n	Usual analysis			Best result*	Exact value
0.1	0.86	1.01	0.92	0.99	0.969
0.2	0.75	0.91	0.83	0.89	0.873
0.5	0.52	0.59	0.60	0.61	0.576
1.0	0.323	0.323	0.365	0.365	0.332

* The term best result applies for small n, not $n = 1$

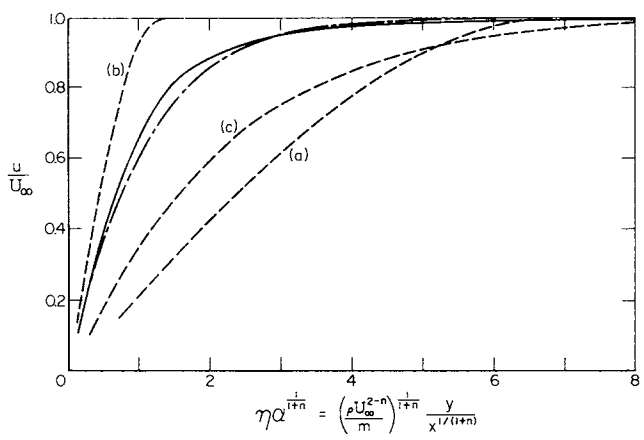

Fig. 2. Velocity profile for $n = 0.1$ —— exact result; — — — — best result, using Equations (8) and (13); - - -, *a:* usual result, Equations (9) and (5), *b:* Equations (9) and (13), *c:* Equations (8) and (5).

CONCLUSIONS

For flow of power-law fluids the velocity profile Equation (8) is a better approximation to the exact solution than is the profile usually assumed for Newtonian fluids, Equation (9). The mean-squared residual can be used as an error criterion: the solution having the smallest residual is the best one. By using the mean-squared residual it is possible to choose between different approximate solutions. Better information on the error can only be obtained by computing successive approximations, but the error criterion is useful in conjunction with the integral method as applied here.

NOTATION

$C(n)$ = shear stress coefficient defined by Equation (14)
m = parameter in constitutive equation for power-law fluid
n = parameter in constitutive equation for power-law fluid
R = residual defined by Equation (6)
$||R||_1$ = mean-squared residual, defined by Equation (10)
$||R||_2$ = norm defined by Equation (11)
u = velocity component in x direction
U_∞ = fluid velocity outside boundary layer
x = distance along plate surface
y = distance normal to the surface
z = distance from center of parallel-plate duct

Greek Letters

α = separation constant determined by Equation (5)
δ = boundary layer thickness, Equation (4)
η = distance, defined by Equation (2)
ρ = density
ϕ = dimensionless velocity defined by Equation (1)

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